Indian Statistical Institute, Bangalore

B. Math. Third Year

First Semester - Combinatorics

Final Semesteral Exam | Duration : 4 hours | Date : Nov 12, 2015

Note: There are six questions. Each carries 25 marks. Answer any four. Full Marks: 100

1. Let $\mathbb{F}$ be a field, and V be a vector space of dimension $n+1$ over $\mathbb{F}$. Let $\{e_1, ..., e_{n+1}\}$ be a basis of V, and put $e_{n+2} = e_1 + ... + e_{n+1}$. Consider the point-set $X = \{[e_1], ..., [e_{n+2}]\}$ in $PG(n, \mathbb{F})$. Show that the point-wise stabilizer of X in the full automorphism group of $PG(n, \mathbb{F})$ is isomorphic to aut $(\mathbb{F})$.

2. Let O be an oval in $PG(2, \mathbb{F}_q), q$ odd prime power. Prove that every inscribed triangle in O is centrally perspective with its mate w.r.t. O.

3. The set of all hyperovals in the projective plane of order 4 is naturally partitioned into three equal classes such that the intersection sizes within each class is 0 or 2, and the intersection sizes between any two classes is 1 or 3. Assuming this, show that the incidence system, whose blocks are the members of one of these classes, is a 2 design. Compute its parameters. Consider the graph whose vertices are the blocks of this design, with disjointness as adjacency. Compute the parameters of this strongly regular graph and hence show that there is a biplane of order nine.

4. Show that there is a unique biplane of order 4 whose Hussain Chains (with respect to all blocks) consist of $2K_3$'s only. Show that this biplane is self-dual and its full automorphism group is transitive on points as well as on blocks. What is the order of this group? Justify your answer.

5. Show that there are infinitely many numbers n for which the complete graph K_n carries no Hussain Chain. Prove that, upto isomorphism, K_5 carries a unique Hussain Chain.

6. (a) A simplex is said to be optimally inscribed in a hypercube if all the vertices of the former are also vertices of the latter. Show that an n-dimensional regular simplex can be optimally inscribed in an n-dimensional hypercube if and only if there is a Hadamard matrix of order $n + 1$.

   (b) Let t be an even number, and $k > t, \lambda > 0$. Then show that there is a $t - (2k + 1, k, \lambda)$ design if and only if there is a $(t + 1) - (2k + 2, k + 1, \lambda)$ design.